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Computing Periodic Solutions and Their Parameter Dependance

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1. Introduction

The simplest way to compute stable periodic solutions of a system of ODE's is to integrate forward in time. If the initial point is within the basin of attraction of a stable periodic solution the trajectory will eventually approach the periodic solution. It is then necessary to detect when the time signal is periodic, and compute its period.

An alternative is to cast the periodic solution as the solution of a two-point boundary value problem, and solve for the periodic solution directly, using Newton's method, or a relaxation scheme. The period of the oscillation is obtained directly, and unstable or marginally stable periodic solutions can be computed.

This technique has become fairly standard in the study of nonlinear systems, where the structure of the unstable periodic solutions is important. In this talk I will briefly outline the two point boundary value problem used, and how to compute the stability of the periodic solutions. Finally I will show an example, periodic solutions in a system of coupled oscillators.

2. Periodic solutions

Consider the autonomous system of ODE's:

$$x' = F(x, \lambda), \qquad x \in \mathbb{R}^n, \qquad \lambda \in \mathbb{R}. \tag{2.1}$$

Aside: If the system that you wish to solve is non-autonomous, with periodic coefficients, say

$$y' = f(y, t, \lambda) \qquad f(y, t + T, \lambda) = f(y, t, \lambda) \tag{2.2}$$

it may be rewritten as an autonomous system by "suspending" the system. For example, let

$$x = \begin{pmatrix} x\_y \\ x\_t \end{pmatrix} = \begin{pmatrix} y \\ t \end{pmatrix} \in \mathbb{R}^n \times [0, T), \tag{2.3}$$

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then the nonautonomous system (2.2) becomes

$$x' = \begin{pmatrix} f(x\_y, x\_t, \lambda) \\ 1 \end{pmatrix} \tag{2.3}$$

We will write a two point boundary value problem for the periodic solution. In order to keep the boundary value problem on a fixed interval time is scaled by the period. A periodic solution of (2.1) of period T satisfies

$$\begin{aligned} x' - F(x, \lambda) &= 0, \\ x(0) - x(T) &= 0, \end{aligned} \tag{2.4}$$

with $t = T\tau$, this is equivalent to

$$\begin{aligned} x_\tau - TF(x, \lambda) &= 0, \\ x(0) - x(1) &= 0. \end{aligned} \tag{2.5}$$

The linearization of (2.5) about a periodic solution $x(\tau)$ is

$$\begin{aligned} \tilde{x}_\tau - TF_x(x, \lambda)\tilde{x}, \\ \tilde{x}(0) - \tilde{x}(1). \end{aligned} \tag{2.6}$$

If a particular x solves (2.5), then differentiating (2.5) with respect to τ gives

$$\begin{aligned} \tilde{x}_{\tau\tau} - TF_x(x, \lambda)\tilde{x}_\tau &= 0, \\ \tilde{x}_\tau(0) - \tilde{x}_\tau(1) &= 0 \end{aligned} \tag{2.7}$$

So the linearized equations (2.6) are singular, with null vector given by the τ derivative of the solution. The geometrical interpretation of this singularity is that time may be shifted by any arbitrary amount, or that the phase of any periodic solution is undetermined.

This indeterminacy is removed by adding a phase constraint. Roughly speaking, the number of unknowns $(x(\tau), T)$ is greater than the number of equations by one (e.g. the period T). An additional constraint is needed to make the counting correct.

Necessary conditions for the phase constraint can be found by use of the "Bordering Lemma".

Bordering Lemma [2]

Let

$$\mathcal{A} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

be a linear operator $\mathcal{A} : \mathbb{B} \times \mathbb{R}^r \to \mathbb{B} \times \mathbb{R}^r$. *If* A *is singular and*

$$\dim\mathcal{N}(A) = \mathrm{codim}\mathcal{R}(A) = r \geq 1,$$

then $\mathcal{A}$ is nonsingular if and only if

$$\dim\mathcal{R}(B) = r, \quad \mathcal{R}(B) \cap \mathcal{R}(A) = 0, \tag{2.8a}$$

$$\dim\mathcal{R}(C^*) = r, \quad \mathcal{N}(A) \cap \mathcal{N}(C^*) = 0. \tag{2.8b}$$

For periodic solutions partition (2.5) into the ODE and phase constraint, and the solution into x and the period T. Thus $r = 1$, and the partitioned form of the linearized system (2.6) is

$$\mathcal{A} = \begin{pmatrix} \frac{\partial}{\partial\tau} - TF_x(x,\lambda) & -F(x,\lambda) \\ p_x(x,T,\lambda) & p_T(x,T,\lambda) \end{pmatrix}. \tag{2.9}$$

Here the phase condition has been written

$$p(x) = 0. \tag{2.10}$$

The first condition (2.8a) is satisfied by all phase constraints: Suppose there is a function $\tilde{x}(\tau)$ and nonzero constant ξ such that

$$\tilde{x}_\tau - TF_x\tilde{x} = \xi TF. \tag{2.11}$$

That is, F is in the range of the linearized ODE. Now,

$$TF = x_\tau, \tag{2.12}$$

so we have that

$$\left(\frac{\partial}{\partial\tau} - TF_x\right)^2 \tilde{x} = \left(\frac{\partial}{\partial\tau} - TF_x\right) \xi x_\tau = 0. \tag{2.13}$$

If zero is a simple eigenvalue of the linear operator $\frac{\partial}{\partial\tau} - TF_x$, then is a contradiction, and so (2.8a) is satisfied.

The second condition (2.8b) is not satisfied for all phase constraints. In the space of period one functions, we have

$$\begin{array}{ll} A = \frac{\partial}{\partial\tau} - TF_x & C^* = p_x \\ \mathcal{N}(A) = \{\alpha x_\tau |\ \alpha \in \mathbb{R}\} & \mathcal{N}(C^*) = \{v |\ p_x.v = 0\} \end{array} \tag{2.14}$$

Condition (2.8b) is satisfied if and only if

$$p_x(x)^* x_\tau = \int_{\tau=0}^{1} p_x(x(\tau)).x_\tau(\tau)\mathrm{d}\tau \neq 0. \tag{2.15}$$

3. Phase Constraints

There are probably as many possible phase constraints as there are problems. Some of them only work for a particular class of problems, while others are designed to work with any problem. For a discussion of some of the possible constraints see [7]. I will discuss two which are for general problems.

The constraint used by Poincaré to establish persistence of periodic solutions in celestial mechanics is

$$p(x) = x^0_\tau(0).(x(0) - x^0(0)). \tag{3.1}$$

Here $x^0(\tau)$ is a periodic function (period 1) that is 'nearby' the periodic solution. This has a very simple geometric interpretation, see Figure 3.1. Condition (2.15) is satisfied if

$$x^0_\tau(0).x_\tau(0) \neq 0, \tag{3.2}$$

This means that the tangent to the reference periodic solution must not be perpendicular to the tangent of the periodic solution. For methods based on parameter continuation, x^0 is the periodic solution at a nearby parameter value.

A more robust phase condition was proposed by [3]. It is

$$p(x) = \int_0^1 x^0_\tau(\tau).(x(\tau) - x^0(\tau))\mathrm{d}\tau. \tag{3.3}$$

This can be seen as choosing the phase so that the distance

$$\int_0^1 |x(\tau + \theta) - x^0(\tau)|^2 \ \mathrm{d}\tau \tag{3.4}$$

is minimized at $\theta = 0$. This is more robust because the inner product (2.15) is

$$\int_0^1 x^0_\tau(\tau).x_\tau(\tau)\mathrm{d}\tau, \tag{3.5}$$

which includes all points on the periodic solution. (Note: In practice almost any phase condition will work.)

Aside: For non-autonomous systems we have that $(x\_t)_\tau(\tau) = T \neq 0$, so the constraint $x\_t(0) = 0$ always satisfies (2.15) (and is natural). Phase constraints aren't usually discussed in connection with non-autonomous systems, probably because of this. A phase constraint like (3.3) using $x^0(\tau) = (0, \tau)$ might also be considered natural.

4. Stability of Periodic Solutions

The theory which deals with the stability of periodic solutions is called Floquet Theory. Almost any advanced book on ODE's will discuss it, for example [8] and [5]. [4] also has a discussion of the theory. Usually these are in connection with non-autonomous systems, or "systems with periodic coefficient's".

Suppose we have a periodic solution of (2.5). The trajectory of a point started nearby, at $x(0)+\epsilon y$ will stay near the periodic solution $x(\tau)$, and eventually return to a neighborhood of $x(0)$. The mapping R which takes a perturbation y around the periodic solution is called the Poincaré return map.

The trajectory of the perturbation $\xi(\tau)$ satisfies

$$\begin{aligned} &\xi' - TF_x\xi - \Delta TF = 0 \\ &\xi(0) = y \\ &p_x\xi = 0, \end{aligned} \tag{4.2}$$

and the return map is $Ry = \xi(1)$.

The return map can be expressed in terms of the fundamental solution matrix Y, which is the solution of the initial value problem

$$\begin{aligned} Y_\tau - TF_x(x(\tau),\lambda)Y &= 0 \\ Y(0) - I &= 0. \end{aligned} \tag{4.1}$$

and a particular solution $\eta(\tau)$ where $\eta(\tau)$ is the solution of

$$\begin{aligned} &\eta' - TF_x\eta = F \\ &\eta(0) = 0, \end{aligned} \tag{4.2}$$

In terms of Y and η,

$$\xi(\tau) = Y(\tau)y + \Delta T\eta(\tau). \tag{4.4}$$

The phase constraint gives

$$\int_0^1 p_x(\tau).Y(\tau)y \; \mathrm{d}\tau + \Delta T \int_0^1 p_x(\tau).\eta(\tau) \; \mathrm{d}\tau = 0. \tag{4.5}$$

So,

$$\Delta T = \frac{-1}{\int_0^1 p_x(\tau).\eta(\tau)\mathrm{d}\tau} \int_0^1 p_x(\tau).Y(\tau)\mathrm{d}\tau \tag{4.6}$$

and

$$Ry = \xi(1) = \left(Y(1) - \frac{\eta(1)}{\int_0^1 p_x(\tau).\eta(\tau)\mathrm{d}t} \int_0^1 p_x(\tau).Y(\tau)\mathrm{d}\tau \right) y. \tag{4.7}$$

When the phase constraint (3.1) is used this has a simple geometric interpretation. Equation (4.5) in that case chooses the perturbation of the period ΔT so that y returns to the plane $x^0(0).\xi(1) = 0$. When other phase constraints are used the interpretation is not as simple. One effect of constructing the return map this way is that while $Y(1)$ always

has an eigenvalue at 1 ($Y(1)x_\tau = x_\tau$), whereas the return map R takes $y = x_\tau$ into 0, so the eigenvalue 1 is moved to 0. This improves the numerical conditioning of locating bifurcation points.

Aside: For non-autonomous systems with phase constraint $x\_t(0) = 0$ a simple calculation gives

$$y\_t + \Delta T = 0. \tag{4.8}$$

So perturbations which have $y\_t = 0$ (i.e. which satisfy the phase constraint) return to $Y(1)y$. Discussions of the stability of periodic solutions of non-autonomous systems usually use the Fundamental Solution matrix for the return map R.

The stability of the periodic solution is determined by the eigenvalues of R, which are called the Floquet multipliers of the periodic solution. If the perturbation y is in the eigendirection v corresponding to a Floquet multiplier λ, the value at return is

$$v \to Rv = \lambda v. \tag{4.9}$$

Floquet multipliers which lie inside the unit circle therefore mean that perturbations in those directions grow smaller ($|v| \to |\lambda||v|$). and multipliers outside the unit circle mean that perturbations grow larger. Multipliers lying on the unit circle signal a neutrally stable direction, and may indicate a bifurcation. There are other necessary conditions for the existence of bifurcating branches. The multiplier must cross the unit circle, not just touch, and there are non-degeneracy conditions on higher derivatives of the system.

If a Floquet multiplier lies on the unit circle at 1 the periodic orbit is neutrally stable to an orbit period $T + \Delta T$. If a Floquet multiplier lies on the unit circle at -1, the periodic orbit is neutrally stable to an orbit of period $2T + 2\Delta T$. This is because "folding" the return map, or iterating it twice, squares the multipliers, and so the folding map has a multiplier at 1. The period doubled branch can usually be found using the initial guess

$$\begin{aligned} &x(\tau) + \epsilon y(2\tau) && \tau \in [0, 1/2) \\ &x(\tau) - \epsilon y(2\tau - 1) && \tau \in [1/2, 1) \end{aligned}$$

The situation for a complex multiplier on the unit circle is a little harder to describe. Basically, if the argument of the multiplier ($\tan^{-1}(Imag(\lambda)/Real(\lambda))$) is rational (say p/q), the periodic orbit is neutrally stable to an orbit with period $qT + q\Delta T$. For example, multipliers at $e^{\pm 2\pi i/3}$ indicates a period tripling bifurcation.

Conjugate pairs at irrational angles indicate the bifurcation of quasiperiodic solutions, or invariant tori. In the return map these are invariant circles.

5. Newton's Method for finding a Periodic Solution

The final system which must be solved is

$$\begin{aligned} x_\tau - TF(x, \lambda) &= 0, \\ x(0) - x(1) &= 0. \\ p(x) &= 0. \end{aligned} \tag{5.1}$$

As an example, let's use a simple box scheme to discretize (2.5) and the phase constraint. The mesh points in time are $\{\tau_i\}$, with corresponding values $\{x_i\}$. The discrete version of (2.5) is then

$$\begin{aligned} x_i - x_{i-1} - (\tau_i - \tau_{i-1})TF((x_i + x_{i-1})/2, \lambda) &= 0, \\ x_0 - x_N &= 0. \\ p(x) &= 0. \end{aligned} \tag{5.2}$$

To implement Newton's method we have to solve the linearization of this, which is

$$\begin{aligned} \tilde{x}_i - \tilde{x}_{i-1} - (\tau_i - \tau_{i-1}) + TF_x(x_{avg}, \lambda)(\tilde{x}_i + \tilde{x}_{i-1})/2 + F(x_{avg}, \lambda)\tilde{T} &= r_i, \\ \tilde{x}_0 - \tilde{x}_N &= s. \\ \sum_i p_{x_i}(x)\tilde{x}_i &= 0. \end{aligned} \tag{5.3}$$

Where, $x_{avg} = (x_i + x_{i-1})/2$. The block structure of this system is

$$\left(\begin{array}{ccccccc|cc} \bullet & & & & & & & \bullet & F_1 \\ \bullet & \bullet & & & & & & & F_2 \\ & \bullet & \bullet & & & & & & F_3 \\ & & \bullet & \bullet & & & & & F_4 \\ & & & \bullet & \bullet & & & & F_5 \\ & & & & \bullet & \bullet & & & F_6 \\ & & & & & \bullet & \bullet & & F_{N-1} \\ \hline I & & & & & & & -I & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \end{array}\right) \left(\begin{array}{c} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \hline \tilde{x}_N \\ \tilde{T} \end{array}\right) = \left(\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ \hline s \\ 0 \end{array}\right) \tag{5.4}$$

If we block the problem into $\{\tilde{x}_i\}_1^{N-1}$ and the pair $\{\tilde{x}_N, \tilde{T}\}$ the upper left block is lower bi-diagonal. One method of solving this is to use block elimination

Block Elimination Algorithm

To solve

$$\begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

when A is nonsingular, let

$$\begin{aligned} Av &= r \\ AZ &= -B \end{aligned} \tag{5.5}$$

Then

$$x = v + Zy. \tag{5.6}$$

and y is the solution of the equations

$$(C^* Z + D)y = s - C^* v. \tag{5.7}$$

For (5.4), A is lower bi-diagonal, so the vector v and the rectangular matrix Z can be found by elimination, which is equivalent to an integration for each column. The first columns of Z are the fundamental solution matrix, and the last column of Z is the particular solution η (eq. 4.3). The Floquet multipliers can therefore be easily found during the bordering algorithm.

The Newton correction is found by solving (5.7), which is a square system of size one greater than the number of components of x_N.

6. Continuation

In the following example we will track the solutions of the two point boundary value problem as a parameter is changed. The particular algorithm used is called Pseudo-arclength continuation [9]. This requires that an additional constraint, called the pseudo-arclength constraint, be added. The resulting system for the periodic solution is

$$\begin{aligned} x_\tau - TF(x,\lambda) &= 0, \\ x(0) - x(1) &= 0. \\ \int_0^1 x_\tau^0(\tau).(x(\tau) - x^0(\tau))\mathrm{d}\tau &= 0 \\ \int_0^1 \dot{x}^0(\tau).(x(\tau) - x^0(\tau))\mathrm{d}\tau &= ds. \end{aligned} \tag{6.1}$$

The derivative of the reference solution with respect to arclength $\dot{x}^0$ is used in the constraint. This derivative is found using the system

$$\begin{aligned} \dot{x}_\tau - TF_x\dot{x} - \dot{T}F - TF_\lambda\dot{\lambda} &= 0, \\ \dot{x}(0) - \dot{x}(1) &= 0. \\ \int_0^1 x_\tau^0(\tau).(\dot{x}(\tau) - x^0(\tau))\mathrm{d}\tau &= 0 \\ \int_0^1 \dot{x}^2(\tau)\mathrm{d}\tau + \dot{\lambda}^2 &= 1. \end{aligned} \tag{6.2}$$

Both (6.2) and Newton's method applied to (6.1) are identical to (5.1) with an additional constraint (pseudo-arclength) and unknown (λ), and may be solved using the bordering algorithm.

The continuation method uses an initial solution and its tangent $(\dot{x}^0, \dot{\lambda}^0)$ to predict the solution at a nearby parameter value

$$(x(s), \lambda(s)) \sim (x^0, \lambda^0) + ds(\dot{x}^0, \dot{\lambda}^0).$$

This guess is then corrected using Newton's method for (6.1).

As the continuation progresses along the solution branch the Floquet multipliers are monitored. When the number of multipliers inside the unit circle changes a bisection algorithm is invoked to locate the parameter value at which a multiplier crossed the unit circle, and a special routine is called to classify the bifurcation and construct a list of tangents at the singular point.

7. An Example, Coupled Pendula

I'll briefly describe some results which were obtained using the above algorithms (although not the same discretization) for the problem of a pair of linearly coupled pendula. For more details see [6], and [1].

The motion of a pair of pendula, with equal damping γ and possibly different applied torques I_1 and I_2, coupled together with a linear torsional spring of strength k is given by

$$\begin{aligned} \ddot{\phi}_1 + \gamma\dot{\phi}_1 + \sin\phi_1 + k(\phi_1 - \phi_2) &= I_1 \\ \ddot{\phi}_2 + \gamma\dot{\phi}_2 + \sin\phi_2 + k(\phi_2 - \phi_1) &= I_2. \end{aligned} \tag{7.1}$$

Here the ϕ's are the angles which the pendula make with the vertical.

This system is a damped Hamiltonian system, with Hamiltonian

$$H(\phi_1,\phi_2,\dot{\phi}_1,\dot{\phi}_2) = \frac{1}{2}((\dot{\phi}_1)^2+(\dot{\phi}_2)^2) - (\cos\phi_1+\cos\phi_2) + \frac{1}{2}k(\phi_1-\phi_2)^2 - I_1\phi_1 - I_2\phi_2. \tag{7.2}$$

System (7.1) can be written

$$\dot{x} = J\nabla_x H - \gamma D x, \tag{7.3}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.4}$$

$$x = \mathrm{col}\Big(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\Big).$$

There are a large variety of motions of the pendula, including regimes where both pendula "tumble" together (large net torque), and regions where one pendulum "sits" while the other "runs" (small coupling). All periodic solutions are "second kind", that is the sum of the two angles $(\phi_1+\phi_2)/2$ increases by at least 2π over a period. The so called "synchronous solution", where the separation between the pendula is almost constant is such that the average of the two angles increase by exactly 2π. Solutions that bifurcate from the synchronous solution by a period doubling have the average angle increase by 4π and so forth.

The following figures show the periodic solutions connected to the synchronous solution at two different values of the difference between the torques $((I_1 - I_2)/2)$. All show the same characteristic shape, at large average torque the two pendula both "tumble" (the

curve of solutions approaches the synchronous solution), and as the torque is decreased the pendula spend more and more time near $\phi = \pi$, and so the period increases, eventually approaching a heteroclinic connection.

Figure 7.1 illustrates a period doubling bifurcation, and 7.2 shows a larger difference between the torques, at which a pair of heteroclinic connections which have appeared. One of this new pair has the structure associated with a Shil'nikov bifurcation, which gives rise to a horseshoe map and its infinite family of periodic solutions. For more details see [6].

8. Conclusion

Not every problem requires that the structure of periodic solutions be computed in detail. However, when that detail is needed it is possible to calculate both stable and unstable periodic solutions, and associated Floquet multipliers, which give detailed information about the stability of the solutions.

The basic technique is to formulate a nonsingular two point boundary value problem, and solve it with Newtons method. The boundary value problem is natural, except that a phase constraint is necessary when working with autonomous systems. Newton's method is considered expensive computationally, but bordering techniques reduce the work to solving a band matrix with small band. The method is therefore not much more expensive than integration techniques.

Finally, this approach is proving useful in the study of dynamical systems. To understand the approach to chaotic motion it is necessary to understand the topology and stability of periodic solutions over a range of parameter values. It has become feasible to compute this type of structure numerically, and so study the transistions in realistic systems.

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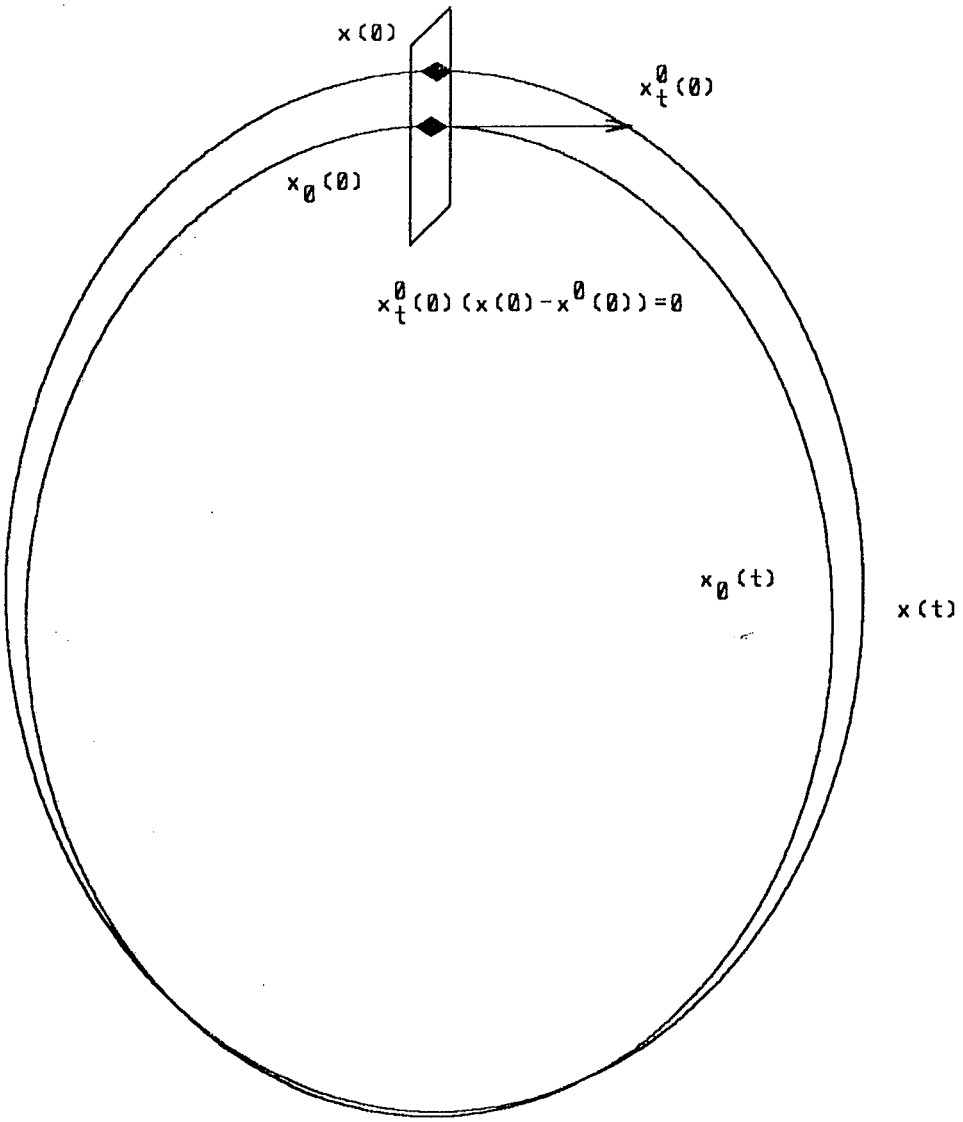

Figure 3.1. The Poincaré phase constraint.

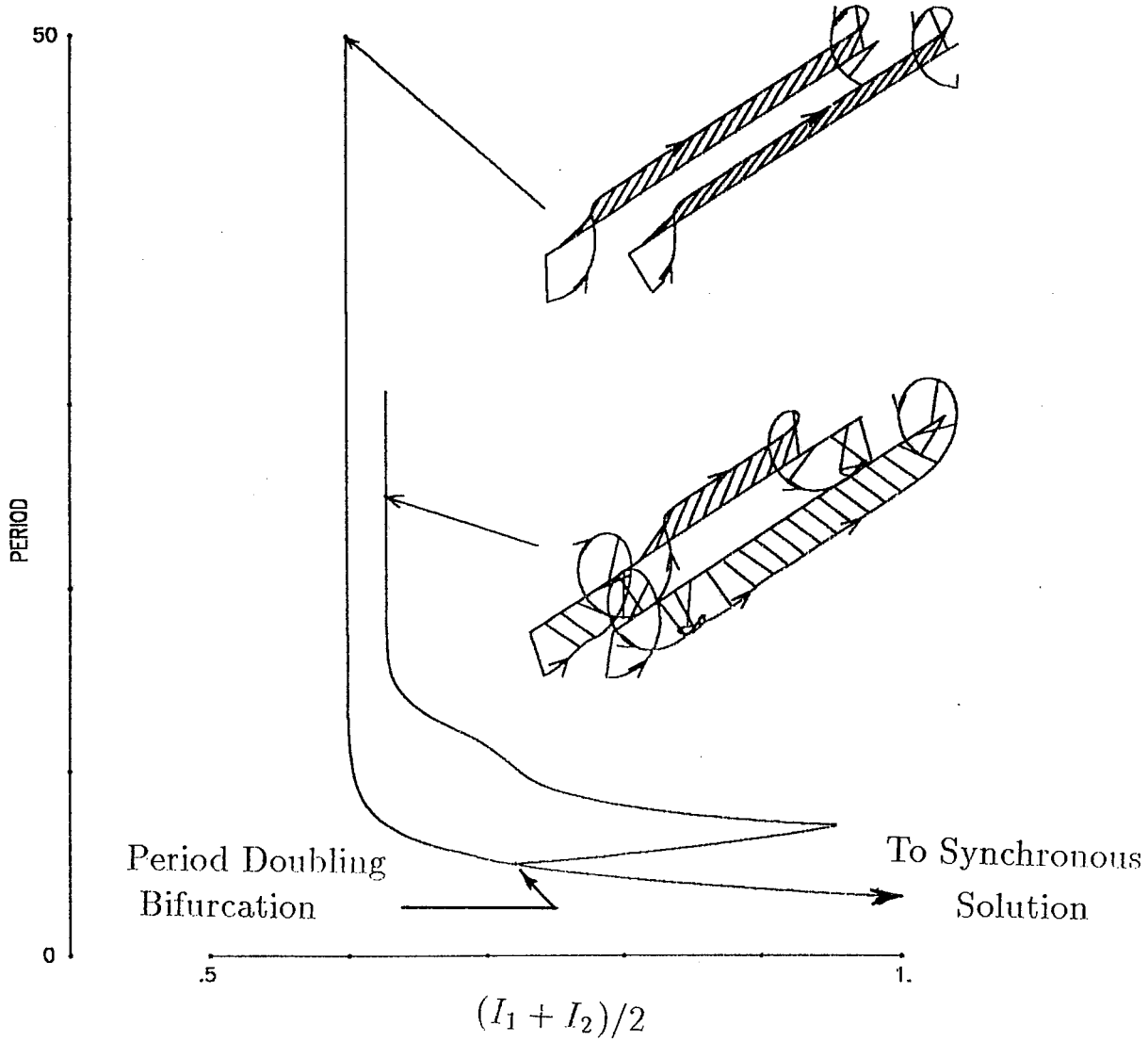

Figure 7.1. Period as a function of average torque I for equal torques. Coupling strength k=.1, and damping $\gamma = .5$. (From [6].)

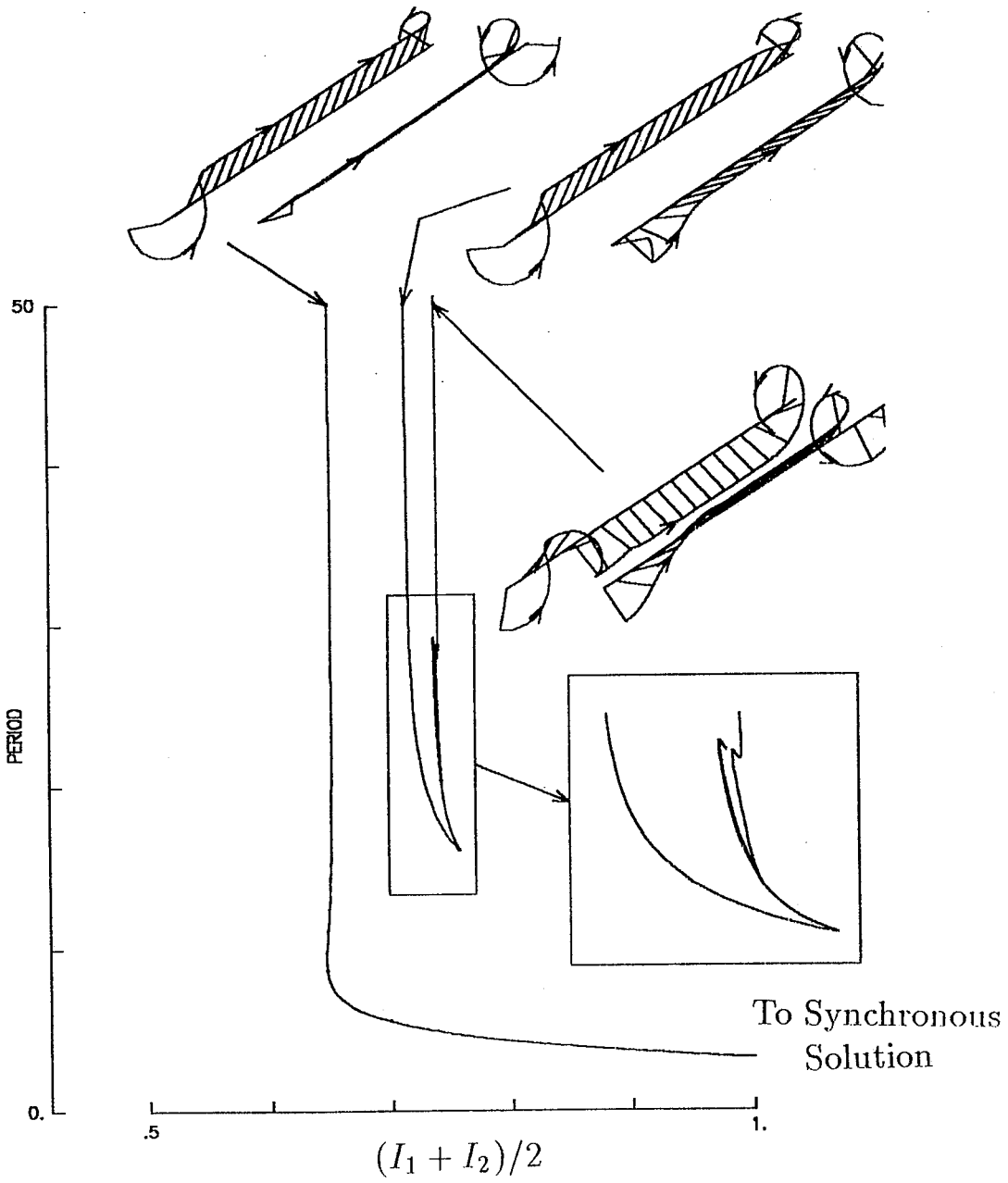

Figure 7.2. Period as a function of average torque I for $I_1 - I_2 = 1.1k\pi$. Coupling strength k=.1, and damping $\gamma = .5$. (From [6].)

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